UDC 532.5 + 539.3

THE STREAM FLOW SCHEME FOR INVESTIGATING THE EQUILIBRIUM FORMS OF ELASTIC PLATES IN A STREAM OF FLUID AND PROBLEMS OF OPTIMIZATION

PMM Vol. 43, No. 1, 1979 pp. 83-90
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(Received January, 23, 1978)

The problem of determining nontrivial equilibrium forms of thin elastic plates in a stream of perfect fluid is considered. The scheme of stream flow with an infinite cavity (the Kirkhoff scheme) is used for determining hydrodynamic forces acting on a curved plate. The ensuing boundary value problem is analyzed, and it is shown that the operator of the problem is self-conjugate and positive definite. An analytic solution of the outer hydrodynamic problem is derived and the fluid reaction on the plate determined.

The determination of equilibrium curved forms reduces to solving an integro-differential equation in eigenvalues. The first eigenvalue (the critical velocity of the oncoming stream) and the related eigenfunction (the curved plate equilibrium form) are obtained by the iteration method. The optimization problem of determining the distribution of the plate thickness for which the nontrivial equilibrium forms obtains at the maximum oncoming stream velocity is then formulated. Optimality conditions are established. The optimization problem is solved for a thin three-layer panel, and it is shown that in this case the optimality condition is not only necessary but, also, sufficient.

Problems of optimization for elastic plates interacting with a perfect fluid were previously considered in [1, 2].

1. Statement of the problem and basic equations. Let us consider the problem of perfect fluid flow past an elastic plate OA (Fig. 1). In the undeformed state the plate lies in a plane normal to the y-axis with its leading edge (point A' in Fig. 1) free and its rear edge $(x = 0,\ y = 0)$ fixed to an absolutely rigid semi-infinite plate OB located on the semiaxis $x > 0$ at $y = 0$. To investigate the nontrivial equilibrium positions of the

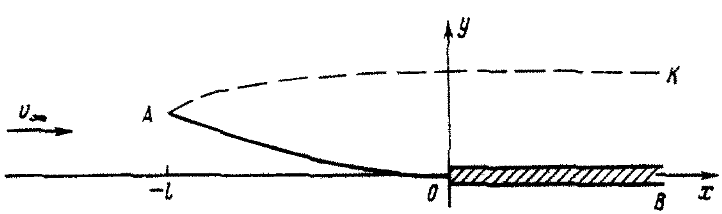

Fig. 1

plate we consider besides its initial undisturbed form $(y = 0)$ some equilibrium curved form OA. Denoting by $u(x)$ $(u \ll l)$ the plate deflections, we write the equation of plate equilibrium and the boundary conditions as

$$(Du_{xx})_{xx} = Q \quad (-l \leqslant x \leqslant 0) \tag{1.1}$$

$$u(0) = u_x(0) = 0, \quad (Du_{xx})_{x=-l} = [(Du_{xx})_x]_{x=-l} = 0 \tag{1.2}$$

Boundary conditions at points $x = 0$ and $x = -l$ define the rigid attachment of the plate rear edge and the absence of bending moments and shear forces at its free edge, respectively.

The cylindrical rigidity denoted by D is related to the [plate] thickness distribution $h(x)$ by formula $D = K_m h^m$, where $K_m = \text{const}$. Parameter m and constant K_m are determined by the plate construction and by constants of its material (Young's modulus and Poisson's ratio). For $m = 3$ and $m = 1$ this formula corresponds to solid and three-layer plates, respectively. In the latter case h is understood to be the thickness of the outer reinforcing layers. The fluid reaction on the plate is denoted by Q.

To determine the reaction of the fluid we consider the hydrodynamic problem of a perfect fluid flow around the contour OA, assuming that the fluid stream flowing past the bent plate becomes separated from the plate and an infinite cavity $BOAK$ is formed. We denote the fluid velocity by $v(x, y)$. At infinity the velocity vector is parallel to the x-axis and its modulus is v_∞. It is assumed that the fluid motion is irrotational and its potential $\varphi(x, y)$ $(v = \nabla\varphi)$ satisfies the Laplace equation.

We represent the potential φ in the form $\varphi = v_\infty x + \Phi$

Function Φ, as well as φ, is harmonic and vanishes at infinity.

$$\Delta\Phi = 0, \quad (\Phi)_\infty = 0 \tag{1.3}$$

Function Φ must satisfy specific boundary conditions at the plate surface and at fluid free surface (the cavity boundary). After linearization the boundary conditions are related to the x-axis, neglecting terms $o(H)$ and $o(U)$, where $H = \max_x h$ and $U = \max_x u$. We draw a zero thickness cut along the semi-infinite interval $x \geqslant -l$ of the x-axis. The upper and lower boundaries of this cut, as well as the functions along the boundaries, are denoted by plus and minus indices, respectively. After linearization we relate to surface S^- the condition of plate surface impermeability to fluid. We obtain

$$(\Phi_y)^- = v_\infty u_x \quad (-l \leqslant x \leqslant 0), \quad (\Phi_y)^- = 0 \quad (x \geqslant 0) \tag{1.4}$$

It is assumed here that the characteristic thickness of the plate is considerably smaller than its characteristic deflection. The kinematic condition at the free surface related to surface S^+ is of the similar form

$$(\Phi_y)^+ = v_\infty f_x \tag{1.5}$$

The dynamic condition $(\nabla\varphi)^2 = \text{const}$ which follows from the Bernoulli integral and implies the constancy of pressure along the cavity boundary assumes the form

$$(\Phi_x)^+ = 0 \quad (x \geqslant -l) \tag{1.6}$$

For a given distribution of deflections $u = u(x)$ the boundary value problem (1.3), (1.4), (1.6) is closed and can be solved for function $\Phi(x, y)$. Having determined function $\Phi(x, y)$, we obtain for the form of the cavity the following quadrature (as implied by (1.5)):

$$f = u(-l) + \frac{1}{v_\infty} \int_{-l}^{x} (\Phi_y(t, 0))^+ dt \qquad (1.7)$$

The fluid reaction distribution on the plate is determined using the Bernoulli integral. After linearization and elementary transformations we obtain

$$Q = p^- - p^+ = -\rho v_\infty (\Phi_x)^- \qquad (1.8)$$

where ρ is the fluid density. In the plate deflection equation (1.1) and boundary conditions (1.2) we represent D in terms of h. We then substitute for Q its expression from (1.8) into Eq. (1.1) and pass in the obtained equation and formulas (1.2) – (1.6) to dimensionless variables: $u' = u/l$, $x' = x/l$, $y' = y/l$, $\Phi' = \Phi/lv_\infty$, and $h' = lh/S$, where S is the cross section area of the elastic plate OA (the primes are henceforth omitted). In this way we obtain for functions u and Φ the closed boundary value problem

$$(h^m u_{xx})_{xx} = -\lambda(\Phi_x)^-, \quad \lambda = \rho v_\infty^2 S^{-m} K_m^{-1} l^{m+3} \quad (-1 \leqslant x \leqslant 0) \qquad (1.9)$$

$$u = u_x = 0 \quad (x = 0), \quad h^m u_{xx} = (h^m u_{xx})_x = 0 \quad (x = -1)$$

$$\Delta\Phi = 0, \quad (\Phi)_\infty = 0 \qquad (1.10)$$

$$(\Phi_x)^+ = 0 \quad (-1 \leqslant x)$$

$$(\Phi_y)^- = u_x \ (-1 \leqslant x \leqslant 0), \ (\Phi_y)^- = 0 \ (x > 0)$$

The hydrodynamic problem (1.10) for the potential Φ and problem (1.9) of bending are interrelated, since the boundary conditions for Φ contain the derivative of the plate deflection distribution, and the derivative of potential Φ appears in the equation of plate deflection.

Problem (1.9), (1.10) is homogeneous, hence it admits the trivial solution $u = \Phi = 0$. The problem of finding nontrivial solutions of this problem leads to a problem in eigenvalues, in which λ represents the eigenvalue. We seek the first eigenvalue λ which corresponds to critical values of parameters ρ, v_∞, l, S, and K_m.

The existence of nontrivial solutions in system (1.9) may be taken as an indication of instability of the undeformed plate position.

2. **Analysis of the boundary value problem.** Since problem (1.10) for potential Φ is linear with respect to u and independent of h, the expression for $(\Phi_x)^-$ in the right-hand side of Eq. (1.9) can be written as

$$(\Phi_x)^- = Lu$$

where L is some linear operator that is self-conjugate and positive. We shall prove that.

Let $u^1(x)$ and $u^2(x)$ be two arbitrary functions that are twice differentiable on segments $[-1, 0]$, and let Φ^1 and Φ^2 be solutions of the boundary value problem (1.10) for $u = u^1$ and $u = u^2$, respectively. We complement the definition of functions u^1 and u^2 on segment $[0, \delta]$, where δ is some positive number, by setting $u^1(x) = u^2(x) \equiv 0$. We obtain

$$\int_{-1}^{0} u^1 Lu^2 dx = \int_{-1}^{\delta} u^1 Lu^2 dx = -\int_{-1}^{\delta} u\,(\Phi_x{}^2)^- dx \qquad (2.1)$$

Applying to the last integral in (2. 1) the formula of integration by parts, taking into account that at $x = -1$ $(\Phi^1)^+ = (\Phi^2)^- = 0$, and using the boundary conditions for potentials Φ^1 and Φ^2, we obtain

$$\int_{-1}^{0} u^1 Lu^2 dx = \int_{-1}^{\delta} u_x{}^1 (\Phi^2)^- dx = \int_{-1}^{\delta} (\Phi_y{}^1)^- (\Phi^2)^- dx = \int_{-1}^{\delta} [(\Phi_y{}^1)^- (\Phi^2)^- - (\Phi_y{}^1)^+ (\Phi^2)^+]\, dx$$

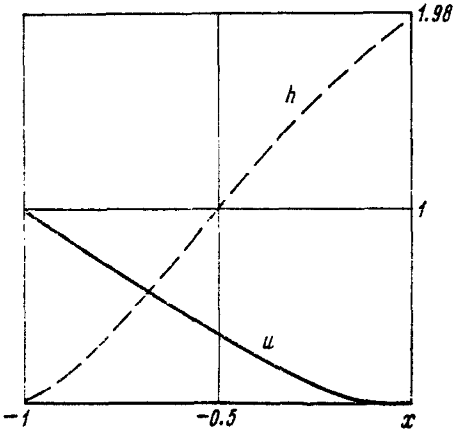

Fig. 2

We denote by $S_\delta{}^+$ and $S_\delta{}^-$ segments of the cut boundaries for which $-1 \leqslant x \leqslant \delta$; by Σ_δ the circle of radius δ with center at point $(-1, 0)$, and by n the inward normal to the boundary $S_\delta = S_\delta{}^+ + S_\delta{}^- + \Sigma_\delta$ (see Fig. 2). In this notation

$$\int_{-1}^{0} u^1 Lu^2 dx = -\int_{S_\delta} \frac{\partial \Phi^1}{\partial n} \Phi^2 ds + \int_{\Sigma_\delta} \frac{\partial \Phi^1}{\partial n} \Phi^2 ds \qquad (2.2)$$

Applying now Green's first formula to the first integral in the right-hand side of (2. 2), we obtain

$$\int_{-1}^{0} u^1 Lu^2 dx = -\int_{S_\delta} \Phi^1 \frac{\partial \Phi^2}{\partial n} d\sigma + \int_{\Sigma_\delta} \frac{\partial \Phi^1}{\partial n} \Phi^2 ds$$

Taking into consideration the boundary conditions for potentials Φ^1 and Φ^2, we rewrite the last equality as

$$\int_{-1}^{0} u^1 Lu^2 dx = \int_{-1}^{\delta} u_x{}^1 (\Phi^2)^- dx + \int_{\Sigma_\delta} \left(\frac{\partial \Phi^1}{\partial n} \Phi^2 - \frac{\partial \Phi^2}{\partial n} \Phi^1\right) ds$$

Integrating by parts and passing to the limit $\delta \to \infty$, we obtain the formula

$$\int_{-1}^{0} u^1 Lu^2 dx = \int_{-1}^{0} u^2 Lu^1 dx \qquad (2.3)$$

Positiveness of operator L is proved in a similar manner. In formula (2. 2) we set $u^1 \equiv u^2$ and $\Phi^1 \equiv \Phi^2$. Applying Green's second formula, we obtain

$$\int_{-1}^{0} u^1 Lu^1 dx = \int_{V_\delta} (\nabla \Phi^1)^2 d\tau + \int_{\Sigma_\delta} \frac{\partial \Phi^1}{\partial n} \Phi^1 ds \qquad (2.4)$$

Region V_δ is bounded by surface S_δ. With δ in formula (2. 4) approaching infinity we have the inequality

$$\int_{-1}^{0} u^1 L u^1\, dx = \int_V (\nabla \Phi^1)^2\, d\tau > 0 \tag{2. 5}$$

The second integral in (2. 5) is taken over region V which represents the exterior of the semi-infinite cut $-1 < x,\ y = 0$.

Since the positiveness and self-conjugacy of the left-hand side of Eq. (1. 9) is known, the boundary value problem in eigenvalues (1. 9), (1. 10) is self-conjugate and positive definite, which proves that the eigenvalues are positive and real.

3. **Determination of fluid reaction and derivation of the integro-differential equation for plate deflections.** The effect of fluid [flow] on bending of the plate is taken into account by the expression in the right-hand side of Eq. (1. 9). To determine the derivative of potential $(\Phi_x)^-$ we consider the external hydrodynamic problem (1. 10), introducing in the analysis the auxilliary function

$$W = \Phi + i\Psi \tag{3. 1}$$

of argument $z = x + iy$ (i is the imaginary unit). Function W is assumed analytic in the plane with the semi-infinite cut $-1 \leqslant x,\quad y = 0$. For the derivative of function $W(z)$ we have the expression $W' = \Phi_x + i\Psi_x$. Using the Cauchy — Riemann equations and boundary conditions we obtain

$$(\Psi_x)^- = -(\Phi_y)^- = -g(x), \quad g = \begin{cases} u_x, & -1 \leqslant x \leqslant 0 \\ 0, & x \geqslant 0 \end{cases} \tag{3. 2}$$

from which with the first of boundary conditions (1. 10) follows that for $-1 \leqslant x$,

$$\operatorname{Re}(W')^+ = 0,\ \operatorname{Im}(W')^- = -ig \tag{3. 3}$$

Thus for the determination of derivative W' of the analytic function W we obtain from (3. 1) the mixed boundary value problem (3. 3) whose solution obtained by Sherman is of the form

$$W' = -\frac{1}{2\pi i (z+1)^{1/4}} \int_{-1}^{0} \frac{(t+1)^{1/4} u_t\, dt}{t - z} + \frac{1}{2\pi i (z+1)^{3/4}} \int_{-1}^{0} \frac{(t+1)^{3/4} u_t\, dt}{t - z} \tag{3. 4}$$

Passing in expressions in the right-hand side of (3. 4) to the limit $z = x + iy \to x - i0\ (0 > y)$ and using the Sokhotskii — Plemel formula, we obtain

$$(\Phi_x)^- = \frac{1}{2\pi (1+x)^{1/4}} \int_{-1}^{0} \frac{(1+t)^{3/4} u_t\, dt}{t - x} + \frac{1}{2\pi (1+x)^{3/4}} \int_{-1}^{0} \frac{(1+t)^{3/4} u_t\, dt}{t - x} \tag{3. 5}$$

where the integrals are to be understood according to the Cauchy meaning of the principal value. In what follows we also use for the unknown quantity the formula

$$(\Phi_x)^- = -\int_{-1}^{0} K(t, x)\, u_t\, dt \tag{3. 6}$$

$$K(t,x) \equiv \frac{1}{2\pi (t-x)} \left[\left(\frac{1+t}{1+x} \right)^{1/4} + \left(\frac{1+t}{1+x} \right)^{3/4} \right]$$

The obtained formula for the fluid reaction is substituted into the equilibrium equation (1. 9) which, then, yields for the distribution of plate deflections the homogeneous integro-differential equation

$$(h^m u_{xx})_{xx} = \lambda \int_{-1}^{0} K(t, x)\, u_t dt \tag{3. 7}$$

Solution of the boundary value problem for Eq. (3. 7) with conditions (1. 9) was obtained numerically for a constant distribution of plate thickness $(h = 1)$ using the method set forth in [3]. The first eigenvalue thus determined is $\lambda = 5.132$. The related distribution of deflections is shown in Fig. 2 by the solid line.

An an illustration we consider a steel plate 1 m wide and 1 cm thick. The critical velocity of motion of such plate in water is

$$v_\infty = \sqrt{\lambda E S^3 / 12 \rho l^6} \simeq 10 \text{ m/s}$$

Below we consider plates of variable thickness and determine the thickness distribution for which the first eigenvalue reaches its maximum.

4. The problem of optimization. Taking into account that the boundary value problem (3. 7) with conditions from (1. 9) is positive definite and self-conjugate, the first eigenvalue λ is determined using the first variational principle of Rayleigh [3]

$$\lambda = \min_u J(h, u) \tag{4. 1}$$

$$J(h, u) = \left(\int_{-1}^{0} h^m u_{xx}^2 dx \right) \Big/ \int_{-1}^{0} \int_{-1}^{0} K(t, x) u(x) u_t(t)\, dt\, dx$$

In this case the minimum is sought in the class of functions that are twice continuously differentiable and satisfy the boundary conditions in (1. 9) formulated for $x = 0$. The other two boundary conditions in (1. 9) are inherent to functional J and, thus, automatically satisfied.

Consider the following problem of optimization: find among all continuous functions $h(x)$ that satisfy the isoperimetric conditions of constancy of the plate section

$$\int_{-1}^{0} h(x)\, dx = 1 \tag{4. 2}$$

a function which maximizes the first eigenvalue λ, i. e.

$$\lambda^* = \max_h \min_u J(h, u) \tag{4. 3}$$

The necessary condition of optimality is of the form

$$h^{m-1} u_{xx}^2 = c^2 \tag{4. 4}$$

where c is the constant Lagrange multiplier which corresponds to the isoperimetric condition (4. 2).

If the dependence of bending rigidity D on thickness h is linear $(m = 1)$ condition (4. 4) is evidently independent of function h. This makes possible the analytical solution of the optimization problem (4.3). In fact, from Eq. (4.4) with boundary conditions (1.9) we find that for $x = 0$ the distribution of deflections $u^*(x)$ of the optimal plate is

$$u^* = cx^2/2 \tag{4.5}$$

Equation (3.7) with boundary conditions (1.9) formulated for $x = -1$ and allowance for formula (4.5) for u^* yield the following Cauchy problem for the second order ordinary differential equation:

$$h_{xx}^* = \lambda^* \int_{-1}^{0} K(t, x)\, t\, dt, \quad h^*(-1) = h_x^*(-1) = 0 \tag{4.6}$$

The optimal thickness distribution obtained by integrating (4.6) is of the form

$$h^* = \lambda^* \int_{x}^{0} \int_{-1}^{0} (x - \eta) K(t, \eta)\, dt\, d\eta \tag{4.7}$$

Using the isoperimetric condition (4.2) for the eigenvalue λ^* we obtain

$$\lambda^* = \Big(\int_{-1}^{0} \int_{x}^{0} \int_{-1}^{0} (x - \eta) K(t, \eta)\, t\, dt\, d\eta\, dx\Big)^{-1} = 7.567 \tag{4.8}$$

The gain obtained by [thickness] optimization over a constant thickness plate is 47.4%. The optimal thickness distribution $h^*(x)$ is shown in Fig. 2 by the dash line.

Let us prove that in the case of $m = 1$ formula (4.4) represents not only the necessary but, also, the sufficient condition of optimality. To do this we estimate the rest $\lambda^* - \lambda$, with λ^*, u^*, and h^* representing the solution of the boundary value problem (3.7), (1.9), (4.4), and λ and u are the eigenvalue and the eigenfunction of problem (3.7), (1.9) that relate to some arbitrary thickness distribution $h(x)$. It is also assumed that h^* and h satisfy the isoperimetric condition (4.2). We have

$$\lambda^* - \lambda = \min_u J(h^*, u) - \min_u J(h, u) \geqslant J(h^*, u^*) - J(h, u^*) =$$
$$\frac{1}{\chi} \int_{-1}^{0} (h^* - h)(u_{xx}^*)^2\, dx, \quad \chi = \int_{-1}^{0} \int_{-1}^{0} K(t, x)\, u_t^*(t)\, u^*(x)\, dx dt$$

but by the optimality condition (4.4) $(u_{xx}^*)^2 = c^2$, hence

$$\lambda^* - \lambda \geqslant \frac{c^2}{\chi} \int_{-1}^{0} (h^* - h)\, dx \tag{4.9}$$

Since functions h^* and h satisfy the isoperimetric condition (4.2), the right-hand side of inequality (4.9) is zero. Hence $\lambda^* \geqslant \lambda$ and in the case of $m = 1$ (4.4) is the sufficient condition of the over-all optimum, and formulas (4.7) and (4.8) yield the unique solution of the problem of optimality.

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Translated by J. J. D.